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THE SOLUTION OF THE CONTACT PROBLEM OF THE THEORY OF ELASTICITY FOR A THICK STRIP WITH ADHESION†

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The contact deformation of a thick elastic strip when there are shear boundary stresses is considered. The existing asymptotic relations, which describe such interaction, are represented in the form of a system of two singular integral equations of the deformation of an elastic half-plane with additive corrections for the finiteness of the width of the strip. A similar form of writing the equations to a large extent formalizes the transfer of the formulations and the methods of solving problems for a half-plane to similar thick-strip problems. The problem of the indentation of a punch into a strip with adhesion is considered as an example.

Contact problems for a thick strip are characteristic for the closeness of the corresponding integral equations and solutions to the equations and solutions of problems of the deformation of an elastic half-plane [1–3], which, to a considerable extent, enables one to formalize the transfer of the formulations and methods of solving problems for a half-plane to similar problems for a thick strip. Below this procedure is also used to solve the problem of the indentation of a punch into a half-plane with adhesion [4].

1. RELATIONS FOR A THICK STRIP

Suppose a strip of width h occupies the region $\{x, y : x \in (-\infty, \infty), y \in [0, h]\}$ and is in the state of plane deformation, due to the application to its upper boundary $y = h$ of stresses $q_1 = \tau_{xy}$, $q_2 = -\sigma_y$, along the contact region $x \in [-a, b]$. The lower boundary $y = 0$ of the strip is assumed to be rigidly connected to an underformable base.

Using a Fourier transformation one can establish the following relation between the stresses q_1 and q_2 and the displacements u and υ of the upper boundary of the strip along the x and y axes [2–6]

$$\begin{aligned} mu(x) &= \int_{-a}^{b} q_1(\xi)k_1\left(\frac{\xi - x}{h}\right)d\xi + \chi\int_{-a}^{b} q_2(\xi)k_{12}\left(\frac{\xi - x}{h}\right)d\xi \\ m\upsilon(x) &= \chi\int_{-a}^{b} q_1(\xi)k_{12}\left(\frac{\xi - x}{h}\right)d\xi - \int_{-a}^{b} q_2(\xi)k_2\left(\frac{\xi - x}{h}\right)d\xi \end{aligned} \tag{1.1}$$

where E is Young's modulus and ν is Poisson's ratio

$$m = \pi E/[2(1-\nu^2)], \quad \chi = (1-2\nu)/[2(1-\nu)]$$

$$\begin{Bmatrix} k_j(z) \\ k_{12}(z) \end{Bmatrix} = \int_0^\infty \begin{Bmatrix} L_j(s)\cos zs \\ L_{12}(s)\sin zs \end{Bmatrix} \frac{ds}{s}, \quad j = 1, 2$$

$$L_j(s) = \frac{2\kappa \operatorname{sh} 2s - (-1)^j 4s}{D(s)}, \quad L_{12}(s) = \frac{2\kappa(\operatorname{ch} 2s - 1) - 8(\kappa - 1)^{-1}s^2}{D(s)}$$

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$$D(s) = 2\kappa \mathrm{ch} 2s + 4s^2 + 1 + \kappa^2, \quad \kappa = 3 - 4\nu$$

where for $|z| < 2$ we have the following asymptotic expressions

$$k_j(z) = -\ln|z| + c_j + O(z^2), \quad j = 1, 2$$
$$k_{12}(z) = \frac{\pi}{2}\mathrm{sign} z + c_{12} z + O(z^3) \tag{1.2}$$

in which the coefficients c_j and c_{12} depend only on κ [3].

If we introduce into consideration the size $l = (a + b)/2$ of the contact area and the small parameter η, then for a thick strip [5]

$$l/h = \eta \ll 1 \tag{1.3}$$

Relation (1.3) denotes that the argument z of the functions $k_j(z)$ and $k_{12}(z)$ is an infinitesimal of the order of η, which, in turn, enables us to use expressions (1.2) to represent them with terms $O(z^2)$ and $O(z^3)$ omitted [2]. Substitution of these expressions into relations (1.1) and subsequent differentiation of the equations obtained with respect to x leads to the equations

$$mu'(x) + \chi\frac{c_{12}}{h}P_2 = -\pi\chi q_2(x) + \int_{-a}^{b}\frac{q_1(\xi)}{\xi - x}d\xi$$
$$m v'(x) + \chi\frac{c_{12}}{h}P_1 = -\pi\chi q_1(x) - \int_{-a}^{b}\frac{q_2(\xi)}{\xi - x}d\xi \tag{1.4}$$

where

$$P_j = \int_{-a}^{b} q_j(x)dx, \quad j = 1, 2 \tag{1.5}$$

Note that Eqs (1.4) differs from the equations for the deformation of an elastic half-plane [7] solely by the presence on the right-hand sides of the additive constants $\chi c_{12} h^{-1} P_j$, which give a correction for the finiteness of the width of the strip.

2. THE PROBLEM OF THE INDENTATION OF A PUNCH INTO A THICK STRIP WITH ADHESION

Suppose a rigid convex punch is indented into a thick strip with adhesion due to the action of a shear load P_1 and a normal load P_2 (see the figure), connected with one another by the loading relation

$$P_1 = N(P_2), \quad P_2 > 0 \tag{2.1}$$

Connecting the system of coordinates with the punch (see the figure) we will assume that during its indentation, the dimensions $a(t) > 0$ and $b(t) > 0$ of the contact area increase monotonically with time t. This enables us to use the quantity α as the time parameter, assuming, in particular, that $b = b(a)$ and $P_j = P_j(a)$.

The boundary conditions for the interaction of the punch with the strip in the case considered have the form

$$u(x, a) = \varphi(x), \quad v(x, a) = g(x); \quad x \in [-a, b] \tag{2.2}$$

where $\varphi(x)$ is a distribution of the tangential displacement of the boundary of the strip over the contact area and $g(x)$ is a function describing the shape of the punch. We pose the following problem: it is required to obtain the distributions of the stresses $q_1(x, a)$ and $q_2(x, a)$ over the increasing contact area $x \in [-a, b]$ and also the unknown functions $\varphi(x)$ and $b(a)$.

We will replace the boundary displacements u and v in Eqs (1.4) by the right-hand sides of conditions (2.2). Then, following the well-known procedure [2], we multiply the first equation by i, the square root

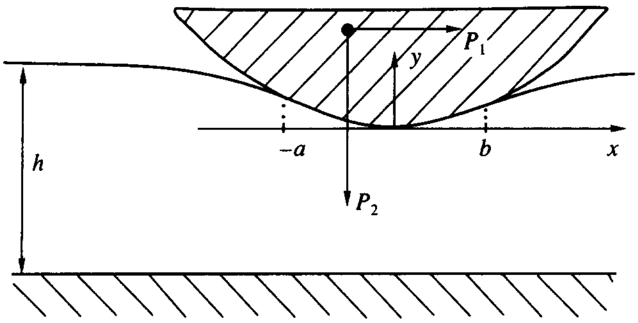

Fig. 1

of –1, and add the result to the second equation. As a result, we obtain the single complex-valued equation

$$-\pi\chi q(x, a) + i\int_{-a}^{b} \frac{q(\xi, a)}{\xi - x} d\xi = \tilde{f}(x, a), \quad x \in [-a, b] \tag{2.3}$$

where

$$q(x, a) = q_1(x, a) + iq_2(x, a), \quad \tilde{f}(x, a) = m[\tilde{g}'(x, a) + i\tilde{\varphi}'(x, a)]$$

$$\begin{Bmatrix} \tilde{\varphi}(x, a) \\ \tilde{g}(x, a) \end{Bmatrix} = \begin{Bmatrix} \varphi(x) \\ g(x) \end{Bmatrix} + \frac{\mu}{m}\begin{Bmatrix} P_2(a) \\ P_1(a) \end{Bmatrix} x, \quad \mu = \chi\frac{c_{12}}{h} \tag{2.4}$$

The condition of equilibrium

$$P(a) \equiv P_1(a) + iP_2(a) = \int_{-a}^{b} q(x, a)dx \tag{2.5}$$

also follows from expressions (1.5), taking the first relation of (2.4) into account.

We further assume that the functions $\varphi'(x)$ and $q'(x)$ satisfy the Hölder condition on any segment of the real axis. Then, by virtue of definitions (2.4), the function $\tilde{f}(x, a)$ will possess the same property with respect to the variable x, and the bounded solution of Eq. (2.3) from the Hölder class $H[-a, b]$ will have the form [8] ($\mathcal{K}$ is an operator)

$$q(x, a) = A^*\tilde{f}(x, a) - \frac{B^*}{\pi i}Z(x, a)\int_{-a}^{b} \frac{\tilde{f}(\xi, a)}{Z(\xi, a)(\xi - x)} d\xi \equiv \mathcal{K}(x, a; \tilde{f}), \quad x \in [-a, b] \tag{2.6}$$

with the condition

$$\int_{-a}^{b} \frac{\tilde{f}(x, a)}{Z(x, a)} dx = 0 \tag{2.7}$$

Here

$$A^* = \chi[\pi(1-\chi^2)]^{-1}, \quad B^* = [\pi(1-\chi^2)]^{-1}$$

$$Z(x, a) = D\sqrt{(a+x)(b-x)}e^{-i\alpha(x, a)}, \quad D = \text{const}$$

$$\alpha(x, a) = \frac{\tau}{2}\ln\frac{a+x}{b-x}, \quad \tau = \frac{1}{\pi}\ln\frac{1+\chi}{1-\chi}$$

Equations (2.1), (2.5), (2.6) and (2.7) represent a system of equations which enable us to obtain the unknown functions $q_1(x, a)$, $q_2(x, a)$, $\varphi'(x)$ and $b(a)$ [4]. By making the formal replacement $\tilde{f}(x, a) \to f(x)$ this system takes the form corresponding to the interaction of the punch with a half-plane, which enables us to use the procedure proposed earlier [4] to simplify it. To do this we must express $\tilde{f}(x, a)$ in condition (2.7) in terms of $\varphi'(x)$ *and* $g'(x)$ using relations (2.4), and substitute expression (2.6) for $q(x, a)$ into the right-hand side of equilibrium condition (2.5), change the order of integration [4] and, as previously, express $\tilde{f}(x, a)$ in terms of $\varphi'(x)$ and $g'(x)$. As a result we obtain the equations

$$\mathcal{K}_1(a; \varphi') = -\omega(a) - i\mu\frac{2}{m}\delta_0\overline{P(a)} \tag{2.8}$$

$$P(a) = P_0(a) - \frac{2}{\pi}\delta_0 B^* ml(1+i\tau)\overline{\omega(a)} - \frac{2}{\pi}\delta_0 B^* m\mathcal{K}_2(a; \varphi') + i\mu\frac{4}{\pi}\delta_0^2 B^* l(1+i\tau)P(a) \tag{2.9}$$

where

$$\mathcal{K}_1(a; \varphi') = \int_{-a}^{b} \frac{e^{-i\alpha(x,a)}}{\sqrt{(a+x)(b-x)}}\varphi'(x)dx, \quad \mathcal{K}_2(a; \varphi') = \int_{-a}^{b} \sqrt{\frac{a+x}{b-x}}e^{i\alpha(x,a)}\varphi'(x)dx$$

$$P_0(a) = \int_{-a}^{b} \mathcal{K}(x, a; mg')dx, \quad \omega(a) = i\mathcal{K}_1(a; g') \tag{2.10}$$

$$\delta_0 = (\pi/2)/\operatorname{ch}(\pi\tau/2), \quad l = l(a) = (a + b(a))/2$$

Note that Eqs (2.8) and (2.9) reduce to the corresponding equations for a half-plane [4] as $h \to \infty$ ($\mu \to 0$). Moreover, we can give it a form that is more convenient for subsequent investigation if we eliminate the unknown function $P(a)$ from their right-hand sides. To do this we must first eliminate $P(a)$ from the right-hand side of Eq. (2.9) using Eq. (2.8), after which, using the equation thus obtained, we can also eliminate $P(a)$ from the right-hand side of Eqs (2.8). As a result, taking into account the equality $B^* = \pi/(4\delta_0^2)$, we obtain the equations

$$\overline{\mathcal{K}_1(a; \varphi')} - i\mu\mathcal{K}_3(a; \varphi') = -\overline{\omega(a)} + i\mu\frac{2}{m}\delta_0 P_0(a), \quad P(a) = P_0(a) + \frac{m}{2\delta_0}\mathcal{K}_3(a; \varphi') \tag{2.11}$$

where

$$\mathcal{K}_3(a; \varphi') = \int_{-a}^{b} \frac{[l(1+i\tau) - (a+x)]e^{i\alpha(x,a)}}{\sqrt{(a+x)(b-x)}}\varphi'(x)dx$$

We must supplement these with loading law (2.1), which relates the real part $P_1(a)$ and the imaginary part $P_2(a)$ of the function $P(a)$. As a result we obtain a system of equations in the functions $\varphi'(x)$ and $b(a)$, which define the contact stresses $q_1(x, a)$ and $q_2(x, a)$ by means of Eq. (2.6) using relations (2.4).

We recall that the calculations carried out above for the problem of the indentation of a punch with adhesion into a thick strip is based on Eqs (1.4) for the deformation of such a strip, the correctness of which, in turn, is ensured by condition (1.3). In the case of an increasing contact area, which occurs in the interaction considered, to satisfy this condition one must limit the size l of the contact area to a certain value l_* and assume

$$l_*/h = \eta \ll 1 \tag{2.12}$$

Condition (2.12) enables us to introduce into consideration an additional small parameter $\varepsilon = \chi c_{12}\eta$ and obtain, taking definition (2.4) for μ into account, the relation

$$\mu = \varepsilon/l_*, \quad |\varepsilon| \ll 1 \tag{2.13}$$

which indicates the fact that $\mu \to 0$ as $\eta \to 0$.

Further, as an example, we will consider the symmetrical formulation of the problem of the indentation of a punch with adhesion into a thick strip

$$N(z) = 0, \quad z \in (-\infty, \infty); \quad g'(x) = R^{-1}x, \quad R = \text{const} \tag{2.14}$$

Conditions (2.14) indicate that, by virtue of the loading relation (2.1), there is no shear load P_1, and the punch has a parabolic form. As far as the solution of this problem is concerned we make the following assumptions

$$b(a) = a, \quad \varphi'(x) \equiv \psi(x) \text{ is an even function.} \tag{2.15}$$

Substituting expression (2.14) for $g'(x)$ into Eq. (2.11) and taking into account assumptions (2.15) we obtain the equations

$$\mathcal{L}(a;\psi) \equiv \mathcal{L}_1(a;\psi) + \mu[\tau a\mathcal{L}_1(a;\psi) - \mathcal{L}_2(a;\psi)] = -R^{-1}\tau\delta_0 a - \frac{1}{2}\mu R^{-1}(1+\tau^2)\delta_0 a^2 \tag{2.16}$$

$$P(a) \equiv P_1(a) + iP_2(a) = i\frac{m}{2R}(1+\tau^2)a^2 + i\frac{m}{\delta_0}[\tau a\mathcal{L}_1(a;\psi) - \mathcal{L}_2(a;\psi)] \tag{2.17}$$

where

$$\mathcal{L}_1(a;\psi) = \int_0^a \frac{\cos\alpha(x,a)}{\sqrt{a^2-x^2}}\psi(x)dx, \quad \mathcal{L}_2(a;\psi) = \int_0^a \frac{x\sin\alpha(x,a)}{\sqrt{a^2-x^2}}\psi(x)dx$$

The functions $P_1(a)$ and $P_2(a)$, defined by equality (2.17), identically satisfy loading relation (2.1) when $N(z) \equiv 0$, and hence we will henceforth consider only Eq. (2.16), which is a Volterra-type integral equation in the unknown function $\psi(x)$.

Equation (2.16) contains the parameter μ, which, by virtue of relation (2.13), can be assumed to be small (apart from a dimensional factor l_*^{-1}). We will point out some properties of this equation, by writing in the general form

$$\mathcal{E}(a;\psi(x)) + \mu\mathcal{F}(a;\psi(x)) = \sum_{n=0}^{\infty}\mu^n f_n(a) \tag{2.18}$$

where we have introduced arbitrary operators that are linear in $\psi(x)$.

We will represent the solution $\psi(x)$ of Eq. (2.18) in the form of a series in powers of μ

$$\psi(x) = \sum_{n=0}^{\infty}\mu^n\psi_n(x) \tag{2.19}$$

in which the functions $\psi_n(x)$ are defined by the recurrence formulae

$$\mathcal{E}(a;\psi_0(x)) = f_0(a); \quad \mathcal{E}(a;\psi_n(x)) + \mathcal{F}(a;\psi_{n-1}(x)) = f_n(a), \quad n \geq 1 \tag{2.20}$$

which are obtained if we substitute series (2.19) into the left-hand side of Eq. (2.18) and equate terms in powers of μ^n in the sums obtained to the term $\mu^n f_n(a)$ from the right-hand side of (2.18).

Suppose that, for a certain $k \geq 1$,

$$f_n(a) = 0, \quad n > k; \quad \mathcal{F}(a;\psi_{k-1}(x)) = f_k(a) \tag{2.21}$$

Then it follows from relations (2.20) that $\psi_n(x) = 0$ for $n \geq k$. In other words, conditions (2.21) lead to termination of series (2.19), making it possible to represent the solution $\psi(x)$ in the form of a finite sum

$$\psi(x) = \sum_{n=0}^{k-1}\mu^n\psi_n(x) \tag{2.22}$$

The functions $\psi_n(x)$, as previously, are defined by formulae (2.20).

We will use these properties of Eq. (2.18) to construct a solution of Eq. (2.16). Comparing Eqs (2.16) and (2.18), we write the operators $\mathscr{E}(a;\psi(x))$, $\mathscr{F}(a;\varphi(x))$ for Eq. (2.16) and, using formulae (2.20), obtain the expressions

$$\psi_n(x) = \frac{1}{R}A^0_{n+1}x^{n+1}, \quad n = 0, 1, 2, \ldots, \quad x \geq 0 \tag{2.23}$$

in which the coefficients A^0_n are found from the recurrence formulae

$$\begin{aligned} A^0_n &= -\frac{1}{\delta_n}(\tau\delta_{n-1} - \gamma_n)A^0_{n-1}, \quad n \geq 3 \\ A^0_1 &= -\tau\frac{\delta_0}{\delta_1}, \quad A^0_2 = -\frac{\delta_0}{2\delta_1\delta_2}[(1-\tau^2)\delta_1 + 2\tau\gamma_2] \end{aligned} \tag{2.24}$$

where

$$\begin{Bmatrix} \gamma_n \\ \delta_n \end{Bmatrix} = \int_0^\infty \frac{\operatorname{th}^n s}{\operatorname{ch} s}\begin{Bmatrix} \sin\tau s \\ \cos\tau s \end{Bmatrix} ds \tag{2.25}$$

Remark 1. The solution of Eq. (2.16) can be obtained by representing it in the form of a power series, by analogy with the approach developed earlier in [9, 10]

$$\psi(x) = \sum_{n=0}^{\infty} A_n x^n, \quad x \geq 0 \tag{2.26}$$

By substituting this series into Eq. (2.16) we can obtain recurrence formulae for A_n and establish that expressions (2.19) and (2.26) are equivalent as solutions of Eq. (2.16).

Remark 2. The quantities defined by the integrals (2.25) satisfy the Poincaré–Perron difference equations

$$\begin{Bmatrix} \gamma_{n+2} \\ \delta_{n+2} \end{Bmatrix} = \frac{(n+1)^2 + n^2 - \tau^2}{(n+1)(n+2)}\begin{Bmatrix} \gamma_n \\ \delta_n \end{Bmatrix} - \frac{(n-1)n}{(n+1)(n+2)}\begin{Bmatrix} \gamma_{n-2} \\ \delta_{n-2} \end{Bmatrix}$$

and are the coefficients of the expansion in a power series of the solution of Heun's class of differential equation [11, 12]. Previously the quantities γ_n and δ_n were obtained when solving the problem of the indentation of a punch of polynomial form with adhesion into an elastic half-plane [9, 10].

The correct use of representation (2.19) for the function $\psi(x)$ requires an investigation of the convergence of the corresponding series, which meets with some difficulties due to the complex form of definition (2.24) of the coefficients A^0_n. The condition for series (2.19) to terminate obtained above enables these difficulties to be avoided, but in this case a small change in the shape of the punch is required. Thus, instead of relation (2.14) we put

$$g'(x) = R^{-1}x + \mu^{k-1}R^{-1}r_k x^k, \quad 3 \leq k \text{ is an odd number} \tag{2.27}$$

where r_k is an arbitrary coefficient, the choice of which is related to satisfying conditions (2.21).

The shape of the punch (2.27) with an odd number k is symmetrical, which enables us, by using assumptions (2.15), as previously, to obtain an equation for $\psi(x)$

$$\begin{aligned} \mathscr{L}(a;\psi) = &-R^{-1}\tau\delta_0 a - \frac{1}{2}\mu R^{-1}(1+\tau^2)\delta_0 a^2 - \\ &-\mu^{k-1}R^{-1}r_k\gamma_k a^k - \mu^k R^{-1}r_k(\delta_{k+1} + \tau\gamma_k)a^{k+1} \end{aligned} \tag{2.28}$$

which differs from Eq. (2.16) by two additional terms on the right-hand side. However, Eq. (2.28), like (2.16), has the form (2.18), and hence results (2.19)–(2.22) hold for it.

The right-hand side of Eq. (2.28) does not contain terms in powers of the parameter μ higher than μ^k, so that the first condition of (2.21) is satisfied for it. The second condition of (2.21) can be satisfied by a corresponding choice of the coefficient r_k. To do this, one must, using formulae (2.20), obtain an expression for the function $\psi_{k-1}(x)$, which contains r_k as a parameter, and substitute this expression into the second equality of (2.21), thereby obtaining a linear equation in r_k with the solution

$$r_k = (k+1)\frac{\delta_k}{\delta_0}(\gamma_{k+1} - \tau\delta_k)A_k^0 \qquad (2.29)$$

The choice of the coefficient r_k from formula (2.29) ensures that the representation of the solution $\psi(x)$ of Eq. (2.28) in the form of finite sum (2.22) is correct. The functions $\psi_n(x)$ corresponding to this representation are determined from formulae (2.20) and have the form

$$\psi_n(x) = \frac{1}{R}\begin{cases} A_{n+1}^0 x^{n+1}, & n = 0, 1, 2, \ldots, k-2 \\ A_k^+ x^k, & n = k-1 \end{cases}, \quad x \geq 0$$
$$A_k^+ = A_k^0 - \frac{\gamma_k}{\delta_k} r_k \equiv (k+1)\frac{\delta_k}{\delta_0}(\delta_{k+1} + \tau\gamma_k)A_0^k \qquad (2.30)$$

Using relation (2.13), expression (2.27) can be written in the form

$$g'(x) = R^{-1}x[1 + \varepsilon^{k-1} r_k (x/l_*)^{k-1}] = R^{-1}x[1 + O(\varepsilon^{k-1})], \quad |\varepsilon| \ll 1$$

which shows that the second term on the right-hand side of Eq. (2.27) can be made as small as desired by choosing the number k to be sufficiently large. Hence, it turns out that a change in the initial shape $g'(x) = R^{-1}x$ of the punch by adding a term that is as small as desired enables one to obtain a solution of the problem of the indentation of a punch with adhesion into a thick strip in close form (2.22), provided the coefficient r_k in Eq. (2.27) is determined from expression (2.29).

Remark 3. The function $\psi_n(x)$ in representations (2.19) or (2.22) for $\psi(x)$, defined by Eqs (2.23) or (2.30) depend on the parameter μ, and hence, as $\mu \to 0$, we have

$$\psi(x) \to \psi_0(x) = R^{-1}A_1^0 x \equiv -R^{-1}\tau\delta_0\delta_1^{-1}x, \quad x \geq 0$$

i.e. the solution for a half-plane [9, 10].

Remark 4. Despite the fact that the formulation and basic equations of the contact problem considered for a thick strip does not contain any fundamental differences from the similar problem for a half-plane, the factor that the width of the strip should be finite ($\mu \neq 0$) leads to the need to find a solution in the form of a series, whereas in the case of a half-plane one must confine oneself to finite sums [9, 10].

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